Evaluating the Risk and Return of an Initially Delta-Hedged Portfolio

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As assumed in the Black Scholes Option Pricing Model, a hedged portfolio consisting of longing delta number of shares of stock and shorting one call option is riskless because the price changes in these two positions offset one another. Such portfolio is a hedged portfolio whose return should be the riskless rate. However, this is true only if the hedged portfolio is continuously rebalanced or adjusted according to price changes. In the real world, such dynamic hedging is rarely implemented because the transaction cost of continuous rebalancing would exceed the benefit of keeping the portfolio riskless. This paper examines the risk and return of such portfolio when it is hedged initially according to delta but is not rebalanced afterward. The result shows that such portfolio, though not riskless, achieves a superior reward to risk ratio than pure stock investment.

Keywords: options, hedging

INTRODUCTION

The option price in the Black Scholes Model is derived by creating a continuously hedged portfolio with the precise number of shares of stock to long to offset one call to create a riskless portfolio. Since such hedged portfolio is riskless, it should earn the riskless rate.

Based on this concept, the price of the call option in the Black Scholes Model was derived by setting the return of this portfolio to the riskless rate. Through this equality, the call option price can be solved by the formula below:

Call option price:

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

where $N(d_1)$ and $N(d_2)$ are cumulative distribution of d_1 and , d_2 respectively.

$N(d_1)$ is the cumulative distribution function (CDF) of the standard normal distribution, evaluated at d_1, which is the probability that a standard normal random variable is less than or equal to d_1.

In the Black Scholes Model, the hedge ratio h or Δ (delta) is equal to the cumulative distribution $N(d_1)$:

- $h = \Delta = N(d_1)$

h represents the number of shares of stock needed to offset the price movement of one short call option to create a hedged portfolio. The value of such portfolio can be written as:

- $V_0 = h\ S_0 - C_0$ (long h shares and short 1 call)

where Vo= initial value of the portfolio
So = initial stock price
Co = initial call option price
h = hedge ratio, the number of shares to long in order to offset 1 call option shorted

The profit for this hedged portfolio from the initial opening date to option expiration date is the difference between the ending value and the initial value of the statically hedged portfolio:

- Initial Date (Today):
- $V_0 = h\ S_0 - C_0$ (long h shares and short 1 call)
- Expiration Date:
- $V_1 = h_1\ S_1 – C_1$
- Profit in dollar = $V_1 - V_0$

For the dynamically hedged portfolio, the delta or the value of h is continuously adjusted according to the newest prices for stock and call option, the stock will always offset the call option precisely, rendering the hedged portfolio riskless.

However, the hedge ratio changes as the stock price and option price change, meaning the portfolio needs to be rebalanced or adjusted every minute according to the newest hedge ratio. In the real world, it is rather expensive to continuously rebalance the portfolio by changing the number of shares to long on a minute-by-minute basis.

Since the equilibrium return of the continuously hedged portfolio is the risk free rate, which means that the profit for this portfolio should be certain.

LITERATURE REVIEW

However, given the prohibitive transaction cost required by continuously rebalancing, such a dynamic hedging strategy is not practical in the real world. Thus, many studies have been conducted to evaluate other alternatives for hedging the portfolio.

Delta hedging is a critical risk management strategy used in derivatives markets to mitigate exposure to price movements. Several studies have examined various aspects of delta hedging, including its effectiveness, transaction costs, and static hedging alternatives.

Hull and White (2017) examine how delta hedging performance varies under different assumptions about asset price dynamics. They show that traditional delta hedging works well when asset prices follow a normal distribution, but its effectiveness deteriorates significantly in the presence of price jumps. The study emphasizes that incorporating more realistic market features, such as jump risk, is crucial for improving hedging accuracy. This study laid the groundwork for subsequent research into improving hedging efficiency.

Xia, K., Yang, X., and Zhu, P. (2023) introduced a two-step delta hedging model for volatility-price elasticity. They incorporated a time-varying negative relationship for implied volatility and underlying price into the delta hedging problem. Their findings indicated that traditional delta hedging methods fail to capture the dynamic relationship between volatility and price changes, leading to suboptimal hedging strategies. Similarly, Strömdahl, N. (2023) compared delta hedging and delta-gamma hedging. The results show that the profit and loss are approximately the same regardless of using delta hedging or delta-gamma hedging. On the one hand, the transaction costs were significantly lower for the delta hedge, but on the other hand, the delta-gamma hedge with the lowest threshold had the most neutral deltas and gammas. The conclusion is therefore that delta-gamma hedging is preferable over delta hedging only if the threshold is low.

The research on delta hedging with stocks with call options has evolved from traditional models to more sophisticated approaches incorporating volatility modeling, transaction cost optimization, and machine learning techniques. While earlier studies emphasized the mechanics of delta hedging, recent research has focused on enhancing hedging performance through statistical modeling, alternative hedging structures, and AI-driven approaches (Qiao & Wan, 2024; Cao et al., 2021).

Advancements in technology have also influenced delta hedging strategies. Cao, Chen, Hull, and Poulos (2021) examined reinforcement learning in hedging and found that artificial intelligence can optimize hedge execution by dynamically adjusting delta based on market behavior. Their study provides evidence that AI-driven hedging strategies outperform traditional Black-Scholes delta hedging, especially in volatile markets (Cao et al., 2021).

Similarly, Qiao and Wan (2024) applied deep learning techniques to delta hedging, showing that machine learning models can optimize hedge ratios more effectively than traditional approaches. Their research highlights that deep learning can adapt to changing market conditions, making delta hedging more dynamic and responsive. However, it does not address the issue of transaction cost. Even though such technological advancement makes delta hedging more dynamic and responsive, it cannot alleviate the high transaction cost problem.

Carr and Wu (2004) explored static hedging as an alternative to delta hedging and found that a portfolio of options can provide similar protection to daily delta hedging but with fewer rebalancing costs. In this paper, static hedging refers to constructing a hedging portfolio at the beginning of the contract period using a fixed set of traded options, and not rebalancing it throughout the life of the hedge. Their study demonstrated that, under certain conditions, static hedging outperforms dynamic delta hedging in minimizing hedging errors. Similarly, Carr and Wu (2013) expanded their research by performing a simulation with options which shows that under continuous price dynamics, discretized static hedges with options perform similarly to the dynamic delta hedge with futures and daily updating, but the static hedges outperform the daily delta hedge when the underlying price process contains random jumps.

Mazzei, G., Bellora, F. G., & Serur, J. A. (2021) addressed the issue of transaction costs in dynamic delta hedging. Their study revealed that frequent rebalancing increases trading costs, which can erode the benefits of hedging. They developed a strategy with neural network that balances transaction costs and hedging accuracy by adjusting the rebalancing frequency based on market conditions. Their model suggests that hedging efficiency can be improved without excessive trading.

Regardless of the techniques used for delta hedging, the transaction cost in dynamic hedging is formidable. In light of the concern over transaction cost in dynamic hedging, this paper examines the risk and return of a purely statically hedging strategy and compare them to those of pure stock investment, which might not have been done by previous work.

In light of this, it would be interesting to see how this initially hedged portfolio perform without continuous rebalancing. That is, keeping the hedge ratio at the initial value all the way through to expiration without rebalancing. Such a portfolio will be referred to as the initially hedged portfolio because it might not be totally hedged after the initial period when the portfolio was established. It will also be referred to as purely statically hedged portfolio. The profit and risk of this portfolio will be compared to those of the corresponding stock investment to assess the relative performance of these two investment approaches.

In this paper, an initially hedged portfolio (as opposed to a continuously hedged portfolio assumed in the Black Scholes Model) consisting delta number of shares and one short position in an actual call option for the S&P ETF is created on February 6, 2025.

- Initial date: Feb 6, 2025
- $V_0 = h\ S_0 - C_0$ (long h shares and short 1 call)
- Expiration Date: April 30, 2025
- $V_1 = h\ S_1 – C_1$
- Where h = $N(d_1)$ = number of shares to long in order to offset 1 call option

- Profit of hedged portfolio in dollar terms = V_1 - V_0

As mentioned, once the hedge ratio is set at the outset, it will be held constant and no adjustment or rebalancing will be made to the portfolio at any time. This purely statically hedged portfolio will be held till option expiration date. The dollar profit for this portfolio is calculated by subtracting the beginning value of the portfolio when it was created from the ending value of the portfolio on option expiration date.

Simulation Procedure for Portfolio With S&P ETF and Option

The profit of this actual portfolio created with S&P ETF stock and call option will be simulated for all possible stock prices. First, the initial hedge ratio is derived by calculating N(d1) with the data from the stock and the call option. This hedge ratio will determine the number of shares to be purchased as one call option is sold short on the same opening date. To simulate the profit of the hedged portfolio, we assume that the stock price of S&P ETF is log normally distributed so that the stock return is normally distributed.

The first variable to be simulated in the standard normal variable Z.

where Z = standard normal distribution with a mean of zero and standard error of one

Z value is simulated by increasing the area by .025 for each increment. Thus, Z values are generated for each percentile on an incremental basis. Each increment of .025 or 2.5 percent probability covers an additional 2.5% probability as compared to the previous Z value.

In this way, the values of Z is generated incrementally with each interval, adding .025 to the cumulative distribution for each successive interval, continuing the same all the way to the highest percentile which is .999 percentile. Altogether, a total of 39 intervals or levels for percentile are generated: .025, .05, .075, .10, .125, ……. .999. Correspondingly, Z values are then generated from these percentiles from the lowest percentile interval to the highest percentile interval, with each interval encompass 2.5% probability.

A Standard normal variable generated for each percentile interval, which is written as $Z_{percentile}$. It is the Z values corresponding to its respective percentile of stock return. The subscript percentile in $Z_{percentile}$ is to indicate which percentile the Z value is corresponding to.

For Z value equal to zero, the stock return from opening date to option expiration date corresponding to Z = 0 would be equal to the mean or the expected return, and the cumulative distribution or percentile would be .5 or 50 percentile. In such case, $Z_{percentile} = 0$ for the case where percentile =.5. For a positive Z value, the stock return outcome corresponding to this Z value would exceed the expected return, and the percentile would be higher than .5. The value for percentile is equal to N(Z), which is the cumulative distribution at Z or the Area to left of Z. Thus, values for $Z_{percentile}$ corresponding to each percentile of stock return can be generated by the inverse function $N^{-1}(Z)$ or the Excel Function: NORM.S.Inv

Following the generation of $Z_{percentile}$ values, all possible values for the stock price growth rate M are derived correspondingly as follows:

M = growth rate of stock price, which is stock return minus dividend yield

M is the simulated growth of stock price on annual basis. It is the hypothetical growth rate of price at each return percentile. Price growth rate is the stock return minus dividend yield on annual basis. (Since the return at each percentile is without variance, the arithmetic and geometric return are equal.)

$$M_{percentile} = E(rs) - q + Z_{percentile}\,\sigma$$

where E(rs) = expected stock return on annual basis
q = annual dividend yield
σ = standard deviation of stock return or implied volatility of stock return
Z percentile = Standard normal variable generated for each percentile interval

$M_{percentile}$ can be interpreted as the annual stock price growth rate at each percentile interval and can be written as:

$$= \text{stock return for the corresponding percentile interval} - \text{dividend yield} + Z_{percentile}\,\sigma$$

Once M values are generated for each percentile interval, the corresponding value of stock on expiration date for each interval can be generated from the lowest percentile interval to the highest percentile interval. Based on the simulated stock price growth rate M, the corresponding stock prices on expiration date are generated at different levels of stock return using the equation

$$S_T = So\ e^{MT}$$

where So = stock price on the initial date the hedged portfolio is established
S_T = Stock price on expiration date for each corresponding stock growth rate (M)
T = number of years to option expiration = days to expiration /365

Likewise, the option price on expiration date for each interval are generated:

$$C_T = Max\ (S_T - K, 0) \text{ where K is the strike price of the option}$$

In this way, all possible stock and option prices on expiration dates are simulated from the lowest stock return percentile to the highest percentile. A total of 39 different stock return percentile intervals are generated. The corresponding stock prices and option prices on expiration date are thus generated for all 39 intervals from this incremental approach.

With these simulated ranges of stock prices on expiration date, the portfolio's value on expiration date at different stock price levels are calculated. Accordingly, the profit or return for the portfolio, which the value of portfolio on expiration date minus the value of portfolio on opening date, are derived for all levels of stock prices.

Thus, the values of the hedged portfolio are calculated at all 39 levels of stock prices. And the profit/loss of this portfolio can then be calculated for all these levels.

Given that the probability of stock return falling within each 2.5% percentile interval is the same throughout (2.5% probability for each range or bracket), the portfolio's expected return can be calculated by taking the average of the portfolio returns at all the stock return ranges. And the standard deviation of portfolio return can then be calculated from the portfolio returns in all the intervals.

The value of the hedged portfolio in the initial period and on expiration date are written respectively as:

$$V_o = h\ S_o - C_o$$

$$V_T = h\ S_T - C_T$$

where h = N(d1) = Δ = hedge ratio = number of shares to long for offsetting short position on one call

The profit for the portfolio is the difference between the ending value and the initial value of portfolio:

$$\text{Profit of portfolio} = V_T - V_o$$
$$= (h\ S_T - C_T) - (h\ S_o - C_o)$$

The actual call bid price is used for the initial call option price since we are selling or shorting the call option at the bid price. The return of the portfolio in percent can be calculated by dividing the profit in dollar by the initial value of the portfolio:

$$\text{Portfolio Return} = \text{Profit of portfolio during option period} / V_o$$

The portfolio return is for the option period, which can be annualized by dividing by T:

$$\text{Annualized Return} = \text{Portfolio Return}/\ T$$

where T = the number of years, which is number of days divded by 365

TABLE 1
DATA FOR CALL OPTION ON S&P ETF FEBRUARY 6, 2025 (NEAR 4PM)

Expiration Date	April 30, 2025
Days to Expiration	83
Expiration period in Year = T	83/365 = .227
Stock Price = So	606
Dividend Yield = q	1.2%
Strike Price =K	600
Call Option (Bid Price) = Co	20.90
Implied Volatility = σ	13.4%
Risk Free Rate	4%

In this example, expected stock return E(rs) for this ETF is assumed to be 12%. Based on the information above, the hedge ratio h or $N(d_1)$ is calculated to be .59, which means that .59 shares of S&P ETF is purchased as one call option is shorted on the initial date to create this hedged portfolio, which will be held till the expiration date. And the hedge ratio .59 will be held constant throughout the option period. The simulated profit for this initially hedged portfolio from the first date to the option expiration date for all percentile intervals are shown in the table below:

TABLE 2
ANNUALIZED RETURN OF INITIALLY HEDGED PORTFOLIO AT VARIOUS STOCK RETURN PERCENTILE

				Initial Hedge Ratio = h = N(d1)						
				h =	0.596619606					
						Vo = h*So-Co =	340.9514813			
Percentile	Z percentile	M_{ann} simulated	ST	CT	VT=h*ST-CT	Profit in $ =VT-Vo	Portfolio Ret	Annual Portfolio Ret	Stock Ret	Annual Stock Ret
0.025	-1.959964	-0.154635174	585.061	0	349.0589527	8.107471384	0.023778959	0.10457012	-0.03455	-0.151948
0.05	-1.6448536	-0.112410386	590.706	0	352.4266896	11.47520831	0.033656426	0.14800717	-0.02524	-0.1109858
0.075	-1.4395315	-0.084897217	594.413	0	354.6385306	13.68704932	0.040143686	0.17653549	-0.01912	-0.084083
0.1	-1.2815516	-0.06372791	597.281	0	356.3498203	15.39833898	0.045162845	0.19860769	-0.01439	-0.0632684
0.125	-1.1503494	-0.046146817	599.674	0	357.7773201	16.82583879	0.049349657	0.21701958	-0.01044	-0.0459055
0.15	-1.0364334	-0.030882074	601.759	1.759273246	357.2621073	16.31062603	0.047838555	0.21037437	-0.007	-0.0307739
0.175	-0.9345893	-0.017234965	603.63	3.62962261	356.507645	15.55616377	0.04562574	0.20064332	-0.00391	-0.0172012
0.2	-0.8416212	-0.004777245	605.342	5.342039956	355.8168895	14.86540818	0.043599776	0.19173395	-0.00109	-0.0047747
0.225	-0.755415	0.006774386	606.934	6.934248425	355.1746238	14.2231425	0.041716031	0.18345002	0.001542	0.00677961
0.25	-0.6744898	0.017618373	608.433	8.432730088	354.5701657	13.61868438	0.039943174	0.17565372	0.004014	0.01765371
0.275	-0.5977601	0.027900143	609.857	9.856938233	353.995668	13.04418674	0.038258191	0.16824385	0.006365	0.02798884
0.3	-0.5244005	0.037730331	611.222	11.22171154	353.4451452	12.49366394	0.036643524	0.16114321	0.008617	0.03789265
0.325	-0.4537622	0.047195867	612.539	12.53874454	352.9138799	11.96239865	0.03508534	0.15429095	0.01079	0.04745003
0.35	-0.3853205	0.056367058	613.818	13.81752935	352.3980432	11.44656193	0.033572407	0.14763769	0.0129	0.05672985
0.375	-0.3186394	0.065302325	615.066	15.06598573	351.8944404	10.94295911	0.032095356	0.14114223	0.01496	0.06578959
0.4	-0.2533471	0.074051488	616.291	16.29089955	351.4003342	10.44885288	0.030646158	0.13476925	0.016982	0.07467848
0.425	-0.1891184	0.082658131	617.498	17.49824017	350.9133166	9.961835351	0.029217751	0.1284877	0.018974	0.08343985
0.45	-0.1256613	0.09116138	618.693	18.69339942	350.4312128	9.479731541	0.027803755	0.12226953	0.020946	0.09211282
0.475	-0.0627068	0.099597292	619.881	19.88138012	349.9520047	9.000523419	0.026398253	0.1160887	0.022907	0.1007337
0.5	2.783E-16	0.108	621.067	21.06695253	349.473768	8.522286753	0.0249956	0.10992041	0.024863	0.1093371
0.525	0.0627068	0.116402708	622.255	22.25479244	348.9946167	8.043135422	0.023590264	0.10374032	0.026823	0.11795696
0.55	0.1256613	0.12483862	623.45	23.44961153	348.5126501	7.561168828	0.022176671	0.09752391	0.028795	0.12662746
0.575	0.1891184	0.133341869	624.656	24.65628958	348.0258998	7.074418561	0.020749048	0.09124581	0.030786	0.13538403
0.6	0.2533471	0.141948512	625.88	25.88001835	347.5322716	6.580790365	0.019301252	0.084879	0.032805	0.14426432
0.625	0.3186394	0.150697675	627.126	27.1264685	347.0294781	6.077996814	0.017826574	0.07839397	0.034862	0.1533095
0.65	0.3853205	0.159632942	628.402	28.40199421	346.514956	5.563474752	0.016317497	0.07175767	0.036967	0.16256567
0.675	0.4537622	0.168804133	629.714	29.71389641	345.9857604	5.034279126	0.014765383	0.0649321	0.039132	0.17208581
0.7	0.5244005	0.178269669	631.071	31.07077553	345.438422	4.486940691	0.013160056	0.05787254	0.041371	0.18193234
0.725	0.5977601	0.188099857	632.483	32.4830224	344.8687493	3.917267992	0.011489224	0.0505249	0.043701	0.19218067
0.75	0.6744898	0.198381627	633.964	33.96352702	344.2715427	3.320061453	0.009737636	0.04282213	0.046144	0.20292432
0.775	0.755415	0.209225614	635.529	35.52874224	343.6401656	2.688684322	0.007885827	0.03467864	0.048727	0.21428269
0.8	0.8416212	0.220777245	637.2	37.20034966	342.9658719	2.014390663	0.005908145	0.0259816	0.051486	0.22641313
0.825	0.9345893	0.233234965	639.008	39.00800272	342.2367001	1.285218861	0.003769507	0.01657675	0.054469	0.23953082
0.85	1.0364334	0.246882074	640.994	40.99412618	341.4355369	0.484055596	0.00141972	0.00624335	0.057746	0.25394362
0.875	1.1503494	0.262146817	643.223	43.22298592	340.5364586	-0.415022723	-0.00121725	-0.00535296	0.061424	0.27011789
0.9	1.2815516	0.27972791	645.8	45.79966938	339.497075	-1.454406315	-0.00426573	-0.01875892	0.065676	0.28881624
0.925	1.4395315	0.300897217	648.916	48.91594188	338.2400317	-2.71144954	-0.0079526	-0.03497226	0.070818	0.31143025
0.95	1.6448536	0.328410386	652.989	52.98855877	336.5972179	-4.354263346	-0.01277092	-0.05616127	0.077539	0.34098421
0.975	1.959964	0.370635174	659.289	59.2886226	334.0558957	-6.895585576	-0.02022454	-0.08893923	0.087935	0.3867022
0.999	3.0902323	0.522091129	682.391	82.39050259	324.7370502	-16.21443103	-0.04755642	-0.20913364	0.126057	0.55434676
							E (r) =	0.09778403		0.11110507
							SD =	0.07818894		0.12793782
							Sharpe Ratio:	0.7390307		0.55577838

where S_0 = stock price on the first date when the hedged portfolio is created
C_0 = call option price on the first date when the hedged portfolio is created
V_0= value of the initially hedged portfolio on the first date when the portfolio created
ST = stock price on option expiration date
CT = option price on expiration date
VT = value of the initially hedged portfolio on option expiration date
E(r) = expected or average return
SD = standard deviation of returns

As seen in the table above, the annualized return for the initial hedged portfolio is variable and is dependent on the stock return. This means that the portfolio is not riskless (as the continuously hedged portfolio in the Black Scholes Model). While the portfolio is not riskless, the return and risk as measured by standard deviation for this initially hedged portfolio is lower than that of the stock.

The Sharpe Ratio is calculated for both the stock and the initially hedged portfolio with the formula below:

Sharpe Ratio = (Expected Return – Risk free rate) /Standard Deviation of Return

As can be seen from the table, the Sharpe Ratio for the initially hedged portfolio is about .739 whereas the Sharpe Ratio for the stock is about .555. This indicates that the initially hedged portfolio, though not riskless, achieves a higher reward-to-risk ratio than stock alone.

CONCLUSION

This result is an interesting finding from this simulation, which implies that covered writers of call options can generate a higher average profit and lower risk than a pure stock investor with a proper hedging strategy that does not require continuous rebalancing. While earlier papers compared static hedging with dynamic hedging, this paper compares static hedging with pure stock investment and found that static hedging achieves a higher reward-to-risk ratio than pure stock investment, which is a significant finding.

REFERENCES

Carr, P., & Wu, L. (2013). Static hedging of standard options. *Journal of Financial Econometrics*, *12*(1), 3–46. https://doi.org/10.1093/jjfinec/nbs014

Cao, J., Chen, J., Hull, J., & Poulos, Z. (2021). Deep hedging of derivatives using reinforcement learning. *The Journal of Financial Data Science*, *3*(1), 10–27. https://doi.org/10.3905/jfds.2020.1.052

Hull, J., & White, A. (2017). Optimal delta hedging for options. *Journal of Banking & Finance*, *82*, 180–190. https://doi.org/10.1016/j.jbankfin.2017.05.006

Mazzei, G., Bellora, F.G., & Serur, J.A. (2021). *Delta hedging with transaction costs: A dynamic multiscale strategy using neural nets*. arXiv preprint arXiv:2109.12337. https://arxiv.org/abs/2109.12337

Qiao, C., & Wan, X. (2024). *Enhancing Black-Scholes delta hedging via deep learning*. arXiv preprint arXiv:2407.19367. https://arxiv.org/abs/2407.19367

Strömdahl, N. (2023). *Delta neutral and gamma neutral hedging for option-based books* (Master's thesis, KTH Royal Institute of Technology). DiVA Portal. Retrieved from https://www.diva-portal.org/smash/get/diva2:1905703/FULLTEXT01.pdf

Xia, K., Yang, X., & Zhu, P. (2023). Delta hedging and volatility-price elasticity: A two-step approach. *Journal of Banking & Finance*, *153*, 106898. Retrieved from https://www.sciencedirect.com/science/article/abs/pii/S0378426623001176?via%3Dihub